

A Study on SEIR Model and Stability Analysis at Disease – Free Equilibrium

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ABSTRACT

In this study, we model an SEIR model, which is an epidemic model. We study the existence of the solution of the model. Also, we study the positive variance of the model. Again, we will find the Basic reproduction number (R_0) which is very important for any disease to know its characteristics (endemic, epidemic and pandemic). We also discuss the disease free-equilibrium (DFE) point and its stability analysis by using Routh - Hurwitz stability criterion.

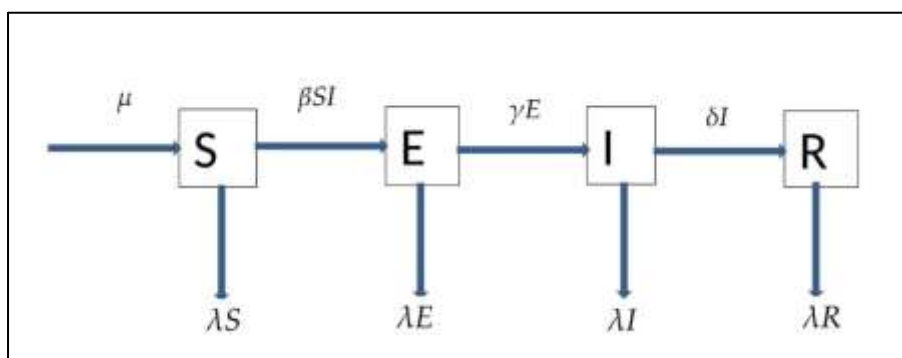
Keywords: *SEIR Model, Basic Reproduction Number, Disease – Free Equilibrium, Stability Analysis, Routh – Hurwitz Stability Criterion.*

Introduction

Epidemic disease is very much contagious from human to human. In the certain criterion, some of the disease attacks most of the susceptible population and spread the disease to make them infective. Infection and infective are two different things. Infection means that a susceptible after getting the disease is infected with the disease but it cannot spread the disease among others. But the infective population means that the person have been infected with the disease and can spread the disease with the susceptible population. In the SEIR epidemic model, the total population is divided into four different compartments S (Susceptible population), E (exposed population), I (Infected population) and R (recovered population). For the most affected prevention, we must know the disease transmission pattern and other important characteristics.

Mathematical model are very important tools for analyzing, planning, implementing the preventive measures. Many of the researchers been applying mathematic model for epidemic disease [1, 2, 3]. Deterministic models are the models in which population is divided into a finite number of subpopulation with the help of ordinary differential equation we can solve and analyses the problem. The epidemic model is not very old. In 1927, the first paper was published by Kermack and McKendrick an SIR model [4]. Many epidemic models for diseases have been discussed by the authors [5, 6, 7].

Mathematical Model



$$\frac{dS}{dt} = \mu - \beta SI - \lambda S$$

$$\frac{dE}{dt} = \beta SI - \gamma E - \lambda E$$

$$\frac{dI}{dt} = \gamma E - \delta I - \lambda I$$

$$\frac{dR}{dt} = \delta I - \lambda R$$

Here, $N = S + E + I + R$, where N is the total population.

Parameters	Symbol
μ	Birth rate
β	Contact rate
γ	Rate for exposed to infective
δ	Recovery rate
λ	Death rate

Now, for positive invariance

The solution $p = \{(S, E, I, R) : N = S + E + I + R\}$ is positively invariant.

Proof –

Since, $N = S + E + I + R$

$$\Rightarrow \frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

$$\Rightarrow \frac{dN}{dt} + \lambda N = \mu$$

This is Linear Differential Equation.

So, I. F. $= e^{\lambda \int dt} = e^{\lambda t}$

$$\Rightarrow N e^{\lambda t} = \int \mu e^{\lambda t} dt + k = \frac{\mu}{\lambda} e^{\lambda t} + k$$

$$\Rightarrow N = \frac{\mu}{\lambda} + k e^{-\lambda t}$$

$$\text{At } t = 0, N(0) = \frac{\mu}{\lambda} + k$$

$$\Rightarrow N = \frac{\mu}{\lambda} + (N(0) - \frac{\mu}{\lambda}) e^{-\lambda t}$$

$$\text{At } t \rightarrow \infty, N(t) \leq \frac{\mu}{\lambda}.$$

So, p is positively invariant.

Again, for positive solution,

Let the initial data is that, the values of S , E , I and R is non – negative at $t = 0$, then the solution is positive for all $t > 0$.

Proof -

Since,

$$\frac{dS}{dt} = \mu - \beta SI - \lambda S \geq -(\beta I + \mu)S$$

$$\Rightarrow \frac{dS}{S} \geq -(\beta I + \mu) dt$$

Integrating this we get,

$$\ln S \geq -(\beta I + \mu)t + \ln c$$

$$\Rightarrow S(t) \geq c e^{-(\beta I + \mu)t}$$

$$\text{At } t = 0, S(0) \geq c$$

$$\Rightarrow S(t) \geq S(0) e^{-(\beta I + \mu)t} \geq 0.$$

Similarly,

$$\frac{dE}{dt} = \beta SI - \gamma E - \lambda E \geq -(\gamma + \lambda)E$$

$$\Rightarrow \frac{dE}{E} \geq -(\gamma + \lambda) dt$$

Integrating this, we get,

$$\ln E \geq -(\gamma + \lambda)t + \ln c$$

$$\Rightarrow E(t) \geq c e^{-(\gamma + \lambda)t}$$

$$\text{At } t = 0, E(0) \geq c$$

$$\Rightarrow E(t) \geq E(0) e^{-(\gamma + \lambda)t} > 0.$$

Again,

$$\frac{dI}{dt} = \gamma E - \delta I - \lambda I \geq -(\delta + \lambda)I$$

$$\frac{dI}{I} \geq -(\delta + \lambda)dt$$

$$\Rightarrow \ln I \geq -(\delta + \lambda)t + \ln c$$

$$\Rightarrow I(t) \geq ce^{-(\delta+\lambda)t}$$

$$\text{At } t = 0, I(0) \geq c$$

$$\Rightarrow I(t) \geq I(0) e^{-(\delta+\lambda)t} > 0$$

And

$$\frac{dR}{dt} = \delta I - \lambda R \geq -\lambda R$$

$$\Rightarrow \frac{dR}{R} \geq -\lambda dt$$

Integrating this, we get,

$$\Rightarrow \ln R \geq -\lambda t + \ln c$$

$$\Rightarrow R(t) \geq ce^{-\lambda t}$$

$$\text{At } t = 0, R(0) \geq c$$

$$\Rightarrow R(t) \geq R(0)e^{-\lambda t} > 0.$$

Hence, all the variables are positive.

Existence of Solution –

Let,

$$k_1 = \mu - \beta SI - \lambda S$$

$$k_2 = \beta SI - \gamma E - \lambda E$$

$$k_3 = \gamma E - \delta I - \lambda I$$

$$k_4 = \delta I - \lambda R$$

$$\left| \frac{\partial k_1}{\partial S} \right| = |-\beta I - \lambda| < \infty, \left| \frac{\partial k_1}{\partial E} \right| = |0| < \infty, \left| \frac{\partial k_1}{\partial I} \right| = |-\beta S| < \infty, \left| \frac{\partial k_1}{\partial R} \right| = |0| < \infty$$

$$\left| \frac{\partial k_2}{\partial S} \right| = |\beta I| < \infty, \left| \frac{\partial k_2}{\partial E} \right| = |-\gamma - \lambda| < \infty, \left| \frac{\partial k_2}{\partial I} \right| = |\beta S| < \infty, \left| \frac{\partial k_2}{\partial R} \right| = |0| < \infty$$

$$\left| \frac{\partial k_3}{\partial S} \right| = |0| < \infty, \left| \frac{\partial k_3}{\partial E} \right| = |\gamma| < \infty, \left| \frac{\partial k_3}{\partial I} \right| = |-\delta - \lambda| < \infty, \left| \frac{\partial k_3}{\partial R} \right| = |0| < \infty$$

$$\left| \frac{\partial k_4}{\partial S} \right| = |0| < \infty, \left| \frac{\partial k_4}{\partial E} \right| = |0| < \infty, \left| \frac{\partial k_4}{\partial I} \right| = |\delta| < \infty, \left| \frac{\partial k_4}{\partial R} \right| = |-\lambda| < \infty.$$

Hence, all these partial derivatives are continuous and bounded. So, there exists a unique solution.

Stability Analysis

For Disease – free equilibrium, we have, the contact rate, $\beta = 0$.

Now, Let

$$\mu - \beta SI - \lambda S = 0 \dots\dots\dots(1)$$

$$\beta SI - \gamma E - \lambda E = 0 \dots\dots\dots(2)$$

$$\gamma E - \delta I - \lambda I = 0 \dots\dots\dots(3)$$

$$\delta I - \lambda R = 0. \dots\dots\dots(4)$$

$$\text{From (1), } \mu = \beta SI + \lambda S = (\beta I + \lambda)S$$

$$\text{If } \beta = 0 \Rightarrow S = \frac{\mu}{\lambda}$$

$$\text{From (2), } \beta SI = (\gamma + \lambda)E$$

$$\text{If } \beta = 0 \Rightarrow E = 0.$$

$$\text{From (3), } \gamma E = (\delta + \lambda)I$$

$$E = 0 \Rightarrow I = 0.$$

$$\text{From (4), } \delta I = \lambda R \Rightarrow R = 0 \text{ as } I = 0.$$

So, $E_0 = \left(\frac{\mu}{\lambda}, 0, 0, 0\right)$ represents the disease – free equilibrium.

For Endemic Equilibrium: $E_1 = (S^*, E^*, I^*, R^*)$

$$\text{From (2), } \beta SI = (\gamma + \lambda)E \dots\dots\dots(5)$$

$$\text{From (3), } (\delta + \lambda)I = \gamma E \dots\dots\dots(6)$$

$$\text{Dividing we get, } \frac{\beta S}{\delta + \lambda} = \frac{\gamma + \lambda}{\gamma}$$

$$\text{So, } S^* = \frac{(\gamma + \lambda)(\delta + \lambda)}{\beta \gamma}$$

Adding (1) and (2), we get,

$$\mu - \lambda S - \gamma E - \lambda E = 0$$

$$\Rightarrow S = \frac{\mu - \gamma E - \lambda E}{\lambda} \dots\dots\dots(7)$$

$$\text{From (3), } (\delta + \lambda)I = \gamma E \Rightarrow I = \frac{\gamma E}{(\delta + \lambda)} \dots\dots\dots(8)$$

$$\text{From (2), } \beta SI = (\gamma + \lambda)E \dots\dots\dots(9)$$

Using (7) & (8) in (9), we get,

$$\beta \frac{\gamma E}{(\delta + \lambda)} \frac{\mu - \gamma E - \lambda E}{\lambda} = (\gamma + \lambda)E$$

Solving this we get,

$$E = \frac{\beta \gamma \mu - \lambda(\gamma + \lambda)(\delta + \lambda)}{\beta \gamma (\gamma + \lambda)}$$

$$= \frac{\mu}{(\gamma + \lambda)} \left(1 - \frac{\lambda(\gamma + \lambda)(\delta + \lambda)}{\beta \gamma \mu} \right)$$

$$\text{Let } R_0 = \frac{\beta \gamma \mu}{\lambda(\gamma + \lambda)(\delta + \lambda)} \dots \dots \dots (10)$$

$$E^* = \frac{\mu}{(\gamma + \lambda)} \left(1 - \frac{1}{R_0} \right) \dots \dots \dots (11)$$

For I, we have from (8),

$$I = \frac{\gamma E}{(\delta + \lambda)} = \frac{\gamma \mu}{(\delta + \lambda)(\gamma + \lambda)} \left(1 - \frac{1}{R_0} \right) \dots \dots \dots (12)$$

$$\text{From (4), } R = \frac{\delta}{\lambda} I = \frac{\delta \gamma \mu}{\lambda(\delta + \lambda)(\gamma + \lambda)} \left(1 - \frac{1}{R_0} \right) \dots \dots \dots (13)$$

Here, E_1 represents an endemic steady state.

Now we establish stability analysis for DFE-

$$\text{Jacobian (J)} = \begin{vmatrix} \frac{\partial k_1}{\partial S} & \frac{\partial k_1}{\partial E} & \frac{\partial k_1}{\partial I} & \frac{\partial k_1}{\partial R} \\ \frac{\partial k_2}{\partial S} & \frac{\partial k_2}{\partial E} & \frac{\partial k_2}{\partial I} & \frac{\partial k_2}{\partial R} \\ \frac{\partial k_3}{\partial S} & \frac{\partial k_3}{\partial E} & \frac{\partial k_3}{\partial I} & \frac{\partial k_3}{\partial R} \\ \frac{\partial k_4}{\partial S} & \frac{\partial k_4}{\partial E} & \frac{\partial k_4}{\partial I} & \frac{\partial k_4}{\partial R} \end{vmatrix} \dots \dots \dots (14)$$

$$= \begin{vmatrix} -\beta I - \lambda & 0 & -\beta S & 0 \\ \beta I & -\gamma - \lambda & \beta S & 0 \\ 0 & \gamma & -\delta - \lambda & 0 \\ 0 & 0 & \delta & -\lambda \end{vmatrix}$$

$$\text{At } E_0, J = \begin{vmatrix} -\lambda & 0 & -\beta \mu / \lambda & 0 \\ 0 & -\gamma - \lambda & \beta \mu / \lambda & 0 \\ 0 & \gamma & -\delta - \lambda & 0 \\ 0 & 0 & \delta & -\lambda \end{vmatrix}$$

So, equilibrium is asymptotically stable if all the eigenvalues are negative.

So the characteristic equation is:

$$\begin{vmatrix} -\lambda - \lambda^* & 0 & -\beta\mu/\lambda & 0 \\ 0 & -\gamma - \lambda - \lambda^* & \beta\mu/\lambda & 0 \\ 0 & \gamma & -\delta - \lambda - \lambda^* & 0 \\ 0 & 0 & \delta & -\lambda - \lambda^* \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - \lambda^*)^2 \{(-\gamma - \lambda - \lambda^*)(-\delta - \lambda - \lambda^*) - \beta\mu\gamma/\lambda\} = 0$$

Solving this we get,

$$\lambda_1 = -\lambda < 0, \lambda_2 = -\lambda < 0, \lambda_3 = -\lambda - \frac{\gamma + \lambda}{2} - \sqrt{(\gamma - \delta)^2 + \frac{4\beta\mu\gamma}{\lambda}} < 0,$$

$$\lambda_4 = -\lambda - \frac{\gamma + \lambda}{2} + \sqrt{(\gamma - \delta)^2 + \frac{4\beta\mu\gamma}{\lambda}} < 0 \text{ if } \beta\mu\gamma < \lambda(\lambda + \gamma)(\lambda + \delta)$$

So, all the eigenvalues are negative, hence the DFE is asymptotically stable.

Conclusion

Here, we have discussed an SEIR model, which is a compartmental model. Here, we have taken that the total population is closed. Here we have calculated the basic reproduction number which is very important to know whether the disease is endemic, epidemic or pandemic. Also, we have discussed the DFE and Endemic equilibrium, and stability analysis by using Routh – Hurwitz equilibrium criterion. If all the eigenvalues are negative then the equilibrium is stable.

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